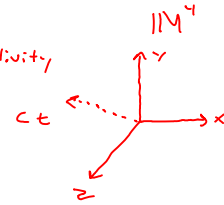
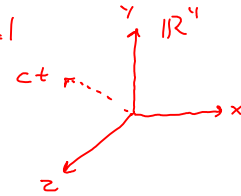


Consider a 4D spacetime labelled by (ct, x, y, z) . "Relativity" is defined w.r.t. $\Lambda \tilde{g} \Lambda^T = g$

i) $g = \eta = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Rightarrow \Lambda$'s include: 3 Rotations (sin, cos) \Rightarrow Special Relativity
3 Boosts (sinh, cosh)



ii) $g = \mathbb{I} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Rightarrow \Lambda$'s include: 6 Rotations (sin, cos) \Rightarrow Purely Spatial



iii) $g_i = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \tilde{g}_i = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Rightarrow \Lambda$'s include: 3 Rotations \Rightarrow Galilean Relativity
3 Boosts

$$\Lambda \tilde{g} \Lambda^T = \tilde{g}$$

Spacetime is much harder to draw! \mathbb{R}^3 bundles over \mathbb{R}^1 .

In SR we have $\underbrace{\Lambda^T \eta \Lambda = \eta}_{O(1,3)}$ which compares to 3D rotations $\underbrace{R^T R = I}_{O(3)}$ orthogonal groups

However we want to restrict to transformations that are continuously connected to the identity. This allows us to build up any transformation by starting w/ the identity and compounding many tiny transformations.

The use of this is 2-fold: a) \bar{I} allows us to do "calculus" w/ the transformations (and eventually leads to Lie algebra structures).
b) These types of transformations give rise to conserved quantities (which we will study later).

However if we just use $O(1,3)$ or $O(3)$ as our condition then these allow

" Λ " = $\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$ "R" = $\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$ these "reflect" a coordinate which is a discrete transformation (not continuously connected to \bar{I})

Note: This is true for any odd # of reflections but not for an even #! Since we can invert two axes with a true rotation!

To eliminate these consider:

$$\det(\Lambda^T \eta \Lambda) = \det \eta \quad \text{or} \quad \det(R^T R) = \det I$$

$$\det \Lambda^T \det \eta \det \Lambda = -1 \quad \det R^T \det R = 1$$

$$(\det \Lambda)^2 = 1 \quad (\det R)^2 = 1 \Rightarrow \det R = \pm 1$$

$$\det \Lambda = \pm 1$$

special means $\det = +1$

In the cases we want to exclude $\det \Lambda = -1$, so we use $SO(1,3)$ or $SO(3)$

Note that $\det \bar{I} = +1$ (a good thing).

There is still one small issue w/ $SO(1,3)$ in particular. One can prove (and you will in your HW) that the upper left term of any Λ , $\begin{pmatrix} \Lambda_{00} & \dots \\ \vdots & \end{pmatrix}$ must satisfy $\Lambda_{00} \geq 1$.

Since $\Lambda_{00} = +1$ for the identity, we need to exclude Λ 's w/ $\Lambda_{00} \leq -1$ (which reverse t!)

The resulting group of transformations is called $SO(1,3)$ or The Proper Orthochronous Lorentz Group

- The complete symmetry group is $\mathbb{I}SO(1,3)^\uparrow = \mathbb{P}^1 \ltimes SO(1,3)^\uparrow$ w/ $4+3+3$ generators, the Poincaré group.
- The algebra of $SO(1,3)^\uparrow$ is interesting: $[R, R] = R$

$$[B, B] = R$$

$$[R, B] = B$$

- Any two inertial observers in SR can be related by one of these transformations ($S \rightarrow S'$).
- To ensure that $SO(1,3)^\uparrow$ is a good symmetry of the rest of our theory, we must always work with objects that transform in a well-defined way (this also makes it easier). So any 3D vector or scalar must be promoted to a 4D vector or scalar w.r.t. $SO(1,3)^\uparrow$, e.g. $\vec{p} \rightarrow \vec{p}^\wedge$
 $t \rightarrow \tau$

Lorentz Transformations and Spacetime Diagrams

When we consider coordinate transformations, it is often useful to draw both the new and old axes together to "visualize" what has changed,



$$\mathbb{R}^2 \quad \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi + y\sin\phi \\ -x\sin\phi + y\cos\phi \end{pmatrix}$$

or

$$x' = x\cos\phi + y\sin\phi$$

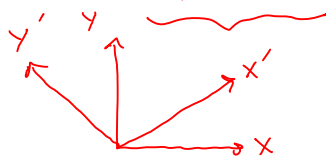
$$y' = -x\sin\phi + y\cos\phi$$

Even though drawing this for rotations is intuitive, we will do it in a slightly more formal way in order to get a method that works for less intuitive boosts.



The x' -axis is where $y' = 0 = -x\sin\phi + y\cos\phi \Rightarrow y = \tan\phi x$

The y' -axis is where $x' = 0 = x\cos\phi + y\sin\phi \Rightarrow y = -\cot\phi x$



} Note that the slopes multiply to -1 (as they should for \perp lines)

Now for Lorentz Boosts



For a boost along x , or a "rotation" in t, x :
$$\begin{pmatrix} ct \\ x \end{pmatrix} \rightarrow \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

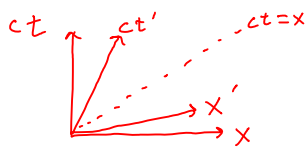
$$= \begin{pmatrix} ct \cosh\phi - x \sinh\phi \\ -ct \sinh\phi + x \cosh\phi \end{pmatrix}$$

$$ct' = ct \cosh\phi - x \sinh\phi$$

$$x' = -ct \sinh\phi + x \cosh\phi$$

The ct' -axis is where $x' = 0 = -ct \sinh\phi + x \cosh\phi \Rightarrow ct = \coth\phi x$

The x' -axis is where $ct' = 0 = ct \cosh\phi - x \sinh\phi \Rightarrow ct = \tanh\phi x$



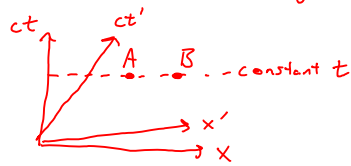
Note that when drawn in \mathbb{R}^2 , the axes are "scissored" under a boost.
Of course if we could draw in M^2 we would see that they are still \perp .

One important feature of this is how to identify lines of constant time or position.

$$ct' = \text{constant} = ct \cosh\phi - x \sinh\phi \Rightarrow ct = \tanh\phi x + \text{constant}$$



Under a boost, a line of constant time no longer remains constant:

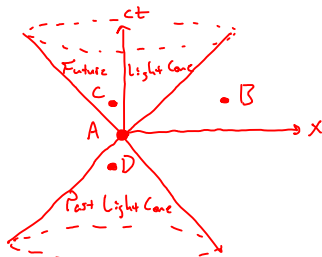


This is the loss of absolute time in SR (as compared to Galilean Relativity), or if you will the notion of absolute simultaneity. This means that it ceases to mean anything to say "A preceded B", but rather now we must say "to some observer A preceded B."

Having given up on an absolute time-ordering of events, we have to rethink causality.

In Galilean Relativity causality is encoded in the statement "causes must precede effects".

In Special Relativity we can view the causal structure using light cones.



Each event in spacetime has a past and future light cone associated with it. The cones for event A are drawn. The surfaces of the cone are connected to A by signals moving at c .

We can conclude:

- A could cause C
 - D could cause A
 - A could not cause B
- } D could cause C